數學思維解題

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第一題

Let ABC be an acute-angled triangle in which BC < AB and BC < CA. Let point P lie on segment AB and point Q lie on segment AC such that P ≠ B, Q ≠ C and BQ= BC= CP. Let T be the circumcentre of triangle AP Q, H the orthocentre of triangle ABC, and S the point of intersection of the lines BQ and CP . Prove that T , H and S are collinear.

翻譯：

設ABC是一個銳角三角形，且滿足BC<AB且BC<CA。設點P位於線段AB上，點Q位於線段AC上，並且P不等於B、Q不等於C，且滿足BQ=BC=CP。設T為三角形APQ的外心，H為三角形ABC的垂心，且S為直線BQ和CP的交點。證明點T、H和S共線。

第二題

Let N= {1, 2, 3, . . .}be the set of all positive integers. Find all functions f : N →N such that for any positive integers a and b, the following two conditions hold:

(1) f (ab) = f (a)f (b), and

(2) at least two of the numbers f (a), f (b) and f (a + b) are equal.

翻譯：

設N= {1, 2, 3, . . .}為所有正整數的集合。找出所有滿足以下兩個條件的函數 f : N →N，使得對於任意正整數a和b，以下兩個條件成立：

（1）f (ab) = f (a)f (b),和

（2）f(a)、f(b)和f(a+b)這三個數中至少有兩個相等。

第三題

Problem 3. An infinite sequence of positive integers $a_1, a_2, \ldots$ is called *good* if

(1) a_1 is a perfect square, and

(2) for any integer $n \geq 2$, a_n is the smallest positive integer such that

$$na_1 + (n-1)a_2 + \ldots + 2a_{n-1} + a_n$$

is a perfect square.

Prove that for any good sequence $a_1, a_2, \ldots$, there exists a positive integer k such that $a_n = a_k$ for all integers $n \geq k$.

題目 3. 若一個無窮正整數序列 $a_1, a_2, \ldots$ 滿足以下條件，則稱其為「良好序列」：

1. a_1 是完全平方數；
2. 對於任何整數 $n \geq 2$，a_n 是最小的正整數，使得

$$na_1 + (n-1)a_2 + \cdots + 2a_{n-1} + a_n$$

   是完全平方數。

證明：對於任何良好序列 $a_1, a_2, \ldots$，存在一個正整數 k，使得對所有 $n \geq k$ 的整數，$a_n = a_k$。

第四題

Given a positive integer $n \geq 2$, determine the largest positive integer N for which there exist $N + 1$ real numbers $a_0, a_1, . . . , a_N$ such that

(1) $a_0 + a_1 = -1/n$, and

(2) $(a_k + a_{k-1})(a_k + a_{k+1}) = a_{k-1} - a_{k+1}$ for $1 \leq k \leq N - 1$.

翻譯：給定一個大於等於二的正整數n，藉由存在$N+1$實數$a_0,a_1,..,a_N$

以(1) $a_0 + a_1 = -1/n$, 和

(2) $(a_k + a_{k-1})(a_k + a_{k+1}) = a_{k-1} - a_{k+1}$ for $1 \leq k \leq N - 1$.

形式來決定最大正整數N.

第五題

For all positive integers n,k, let f(n,2k) be the number of ways an $n \times 2k$ board can be fully covered by nk dominoes of size 2×1. (For example, f(2,2) = 2 and f(3,2) = 3.) Find all positive integers n such that for every positive integer k, the number f(n,2k) is odd.

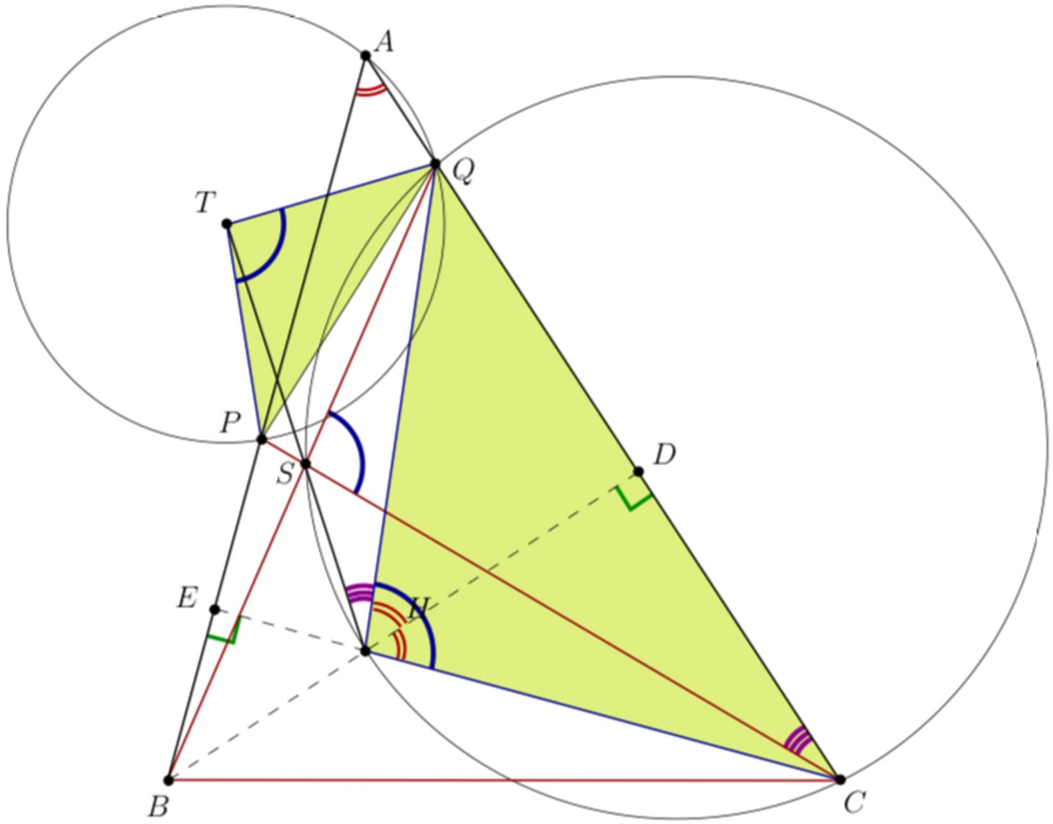
對所有正整數n,k，令f(n,2k)為一個$n \times 2k$ 的方陣可被nk個大小為2×1的骨牌完整覆蓋的方法數(舉例來說，f(2,2) = 2 和 f(3,2) = 3.) 。對所有正整數k，請找出所有正整數n使得f(n,2k) 為奇數。

第六題

Let ABCD be a cyclic quadrilateral with circumcentre O. Let the internal angle bisectors at A and B meet at X, the internal angle bisectors at B and C meet at Y, the internal angle bisectors at C and D meet at Z, and the internal angle bisectors at D and A meet at W. Further, let AC and BD meet at P. Suppose that the points X,Y,Z,W,O and P are distinct. Prove that O,X,Y,Z and W lie on the same circle if and only if P,X,Y,Z and W lie on the same circle.

設 ABCD 為外圓圓心O的內接四邊形。令A和B的內角平分線交於 X，B和C的內角平分線交於Y，內角C和D處的平分線在 Z 處相交，D和A處的內角平分線在W處相交。AC和BD 在 P 處相交。假設點X,Y,Z,W,O,P不重疊。證明O,X,Y,Z和W在同一個圓上當且僅當P,X,Y,Z和W在同一個圓上。

第一題圖



我們要講解的是第一題

In the same way as in the previous solution, we see that ∠PSQ=180度-2∠A, so ∠CSQ=2∠A. From the cyclic quadrilateral AEHD (with E and D feet of the altitudes CH and BH) we see that ∠DHC = ∠DAE = ∠A. Since BH is the perpendicular bisector of CQ, we have ∠DHQ = ∠A as well, so ∠CHQ = 2∠A. From ∠CHQ = 2∠A =∠CSQ, we see CHSQ is a cyclic quadrilateral. This means ∠QHS = ∠QCS.

Since triangles PTQ and CHQ are both isosceles with apex 2∠A, we get ▵PTQ ~ ▵CHQ. We see that one can be obtained from the other by a spiral similarity centered at Q, so we also obtain ▵QTH ~ ▵QPC. This means that ∠QHT =∠QCP. Combining this with ∠QHS = ∠QCS, we see that ∠QHT =∠QCP = ∠QCS = ∠QHS. So ∠QHT = ∠QHS, which means that T , S and H are collinear.

我們可以得到∠PSQ=180度-2∠A因此，∠CSQ=2∠A。從四邊形AEHD(其中E和D分別是高線CH和BH的底點）是圓周四邊形，我們可以得出∠DHC=∠DAE=∠A。因為BH是CQ的垂直平分線，所以有∠DHQ=∠A。因此∠CHQ=2∠A=∠CSQ，我們可以看到CHSQ是一個圓周四邊形。這表示∠QHS=∠QCS。

由於三角形PTQ和CHQ都是頂角為2∠A的等腰三角形，我們可以得到∆PTQ～∆CHQ。我們可以看到其中一個可以通過以Q為中心的旋轉相似變換得到另一個，因此我們也可以得到∠QHT =∠QCP = ∠QCS = ∠QHS，我們得到∠QHT=∠QHS，表示T、S和H共線。

延伸題

設ABC是一個鈍角三角形，且滿足BC<AB且AC<AB。設點P位於直線BC上，點Q位於直線AC上，並且P不等於B、Q不等於C，且滿足BQ=AB=CP。設T為三角形APQ的外心，H為三角形ABC的垂心，且S為直線BQ和CP的交點。證明點T、H和S不共線。